Strong immersions and maximum degree

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Abstract

A graph H is *strongly immersed* in G if G is obtained from H by a sequence of vertex splittings (i.e., lifting some pairs of incident edges and removing the vertex) and edge removals. Equivalently, vertices of H are mapped to distinct vertices of G (*branch vertices*) and edges of H are mapped to pairwise edge-disjoint paths in G, each of them joining the branch vertices corresponding to the ends of the edge and not containing any other branch vertices. We show that there exists a function $d\colon N \to N$ such that for all graphs H and G, if G contains a strong immersion of the star $K_{1,d(\Delta(H))|V(H)|}$ whose branch vertices are $\Delta(H)$-edge-connected to one another, then H is strongly immersed in G. This has a number of structural consequences for graphs avoiding a strong immersion of H. In particular, a class $\mathcal{G}$ of simple 4-edge-connected graphs contains all graphs of maximum degree 4 as strong immersions if and only if $\mathcal{G}$ has either unbounded maximum degree or unbounded tree-width.

In this paper, graphs are allowed to have parallel edges and loops, where each loop contributes 2 to the degree of the incident vertex. A graph without parallel edges and loops is called *simple*.

Various containment relations have been studied in structural graph theory. The best known ones are *minors* and *topological minors*. A graph H is a *minor* of G if it can be obtained from G by a sequence of edge and vertex removals and edge contractions. A graph H is a *topological minor* of G if a subdivision of H is a subgraph of G, or equivalently, if H can be obtained from G by a sequence of edge and vertex removals and by suppressions of vertices of degree two. In a fundamental series of papers, Robertson and Seymour developed the theory of graphs avoiding a fixed minor, giving a description of their structure [14] and proving that every proper minor-closed class of graphs is characterized by a finite set of forbidden minors [15]. The topological minor relation is somewhat harder to deal with (and in particular, there exist proper topological minor-closed classes that are not characterized by a finite set of forbidden topological minors), but a description of their structure is also available [7, 5].

In this paper, we consider a related notion of a graph immersion. Let H and G be graphs. An *immersion* of H in G is a function θ from vertices and edges of H such that

- $\theta(v)$ is a vertex of G for each $v \in V(H)$, and $\theta \restriction V(H)$ is injective.
- $\theta(e)$ is a connected subgraph of G for each $e \in E(H)$, and if $f \in E(H)$ is distinct from e, then $\theta(e)$ and $\theta(f)$ are edge-disjoint.
- If $e \in E(H)$ is incident with $v \in V(H)$, then $\theta(v)$ is a vertex of $\theta(e)$, and if e is a loop, then $\theta(e)$ contains a cycle passing through $\theta(v)$.

An immersion θ is *strong* if it additionally satisfies the following condition:

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- If $e \in E(H)$ is not incident with $v \in V(H)$, then $\theta(e)$ does not contain $\theta(v)$.

When we want to emphasize that an immersion does not have to be strong, we call it *weak*. Let $E(\theta)$ denote $\bigcup_{e\in E(H)} E(\theta(e))$ and let $V(\theta)$ denote $\bigcup_{e\in E(H)} V(\theta(e))$. Let us note that by choosing the subgraphs $\theta(e)$ as small as possible, we can assume that $\theta(e)$ is a path with endvertices $\theta(u)$ and $\theta(v)$ if e is a non-loop edge of H joining u and v, and that $\theta(e)$ is a cycle containing $\theta(v)$ if e is a loop of H incident with v; we call an immersion satisfying these constraints *slim*.

If H is a topological minor of G, then H is also strongly immersed in G. On the other hand, an appearance of H as a minor does not imply an immersion of H, and conversely, an appearance of H as a strong immersion does not imply the appearance as a minor or a topological minor. Nevertheless, many of the results for minors and topological minors have analogues for immersions and strong immersions. For example, any simple graph with minimum degree at least $200k$ contains a strong immersion of the complete graph K_k (DeVos et al. [3]), as compared to similar sufficient minimum degree conditions for minors ($\Omega(k\sqrt{\log k})$, Kostochka [9], Thomason [17]) and topological minors ($\Omega(k^2)$, Bollobás and Thomason [1], Komlós and Szemerédi [8]). A structure theorem for weak immersions appears in DeVos et al. [4] and Wollan [18]. Furthermore, every proper class of graphs closed on weak immersions is characterized by a finite set of forbidden immersions [16].

Chudnovsky et al. [2] proved the following variation on the grid theorem of Robertson and Seymour [13].

Theorem 1. *For every $g \geq 1$, there exists $t \geq 0$ such that every 4-edge-connected graph of tree-width at least t contains the $g \times g$ grid as a strong immersion.*

The variant of Theorem 1 for weak immersions was also proved by Wollan [18]. Note that unlike the grid theorem for minors [13], Theorem 1 does not admit a weak converse—there exist graphs of bounded tree-width containing arbitrarily large grids as strong immersions. A connected graph H with at least three vertices is a *multistar* if it has no loops and contains a vertex c incident with all its edges. The vertex c is the *center* of the multistar and all its other vertices are *rays*. We write $c(H)$ for the center of the multistar and $R(H)$ for the set of its rays. Let the multistar with n rays of degree k be denoted by $S_{n,k}$. Note that every graph with at most n vertices and with maximum degree at most k is contained as a strong immersion in $S_{n,k}$, which has tree-width 1. Furthermore, subdividing each edge of $S_{n,k}$ results in a simple graph of tree-width 2 containing every graph with at most n vertices and with maximum degree at most k as a strong immersion.

Consequently, to turn Theorem 1 into an approximate characterization, we need to deal with the star-like graphs. The main result of this paper essentially states that if the maximum degree of H is k and a k-edge-connected graph G contains a sufficiently large star as a strong immersion, then G also contains H as a strong immersion. Let us now state the result more precisely.

A *k-system of magnitude d* in a graph G is a pair (H, σ), where H is a multistar and σ is a strong immersion of H in G satisfying the following conditions:

(s1) H has at least d edges,

(s2) rays of H have degree at most k (in H), and

(s3) for each $v \in R(H)$, there exists no edge cut in G of size less than k separating $\sigma(c(H))$ from $\sigma(v)$.

Let us remark that by Menger's theorem, **(s3)** implies that G contains k pairwise edge-disjoint paths from $\sigma(c(H))$ to $\sigma(v)$ (not necessarily belonging to or disjoint with the immersion).

A strong immersion θ of $S_{n,k}$ in a graph G *respects* a strong immersion σ of a multistar H in G if $\theta(c(S_{n,k})) = \sigma(c(H))$ and $\theta(R(S_{n,k})) \subseteq \sigma(R(H))$. Let us define $d(k) = (2k+1)^{8k+4}k^2(k+1)$.

Theorem 2. *If $k \geq 3$ and $n \geq 2$ are integers and (H, σ) is a k-system of magnitude at least $d(k)n$ in a graph G, then G contains $S_{n,k}$ as a strong immersion respecting (H, σ).*

If G is k-edge-connected and contains a vertex c with at least d distinct neighbors, then the neighborhood of c contains a k-system of magnitude d. Let us recall that if a graph F has n vertices and maximum degree at most k, then F is strongly immersed in $S_{n,k}$.

Furthermore, the relation of strong immersion is transitive. If H_1 has an immersion θ_1 in H and H has an immersion θ in G, then let $\theta \circ \theta_1$ be defined as follows: $(\theta \circ \theta_1)(v) = \theta(\theta_1(v))$ for each $v \in V(H_1)$ and $(\theta \circ \theta_1)(e) = \bigcup_{f \in E(\theta_1(e))} \theta(f)$ for each $e \in E(H_1)$. Note that $\theta \circ \theta_1$ is an immersion of H_1 in G, and if θ_1 and θ are strong, then $\theta \circ \theta_1$ is strong. Consequently, Theorem 2 has the following corollary.

Corollary 3. *For every integer $k \geq 3$ and a graph F of maximum degree at most k, if a k-edge-connected graph G contains a vertex with at least $d(k)|V(F)|$ distinct neighbors, then F appears in G as a strong immersion.*

The version of Corollary 3 for weak immersions was previously obtained by a different method by Marx and Wollan [12]. As a consequence of Corollary 3, we obtain the following strengthening of Theorem 1.

Theorem 4. *Let $\mathcal{G}$ be a class of 4-edge-connected simple graphs. The following propositions are equivalent:*

(i) There exists a graph F of maximum degree 4 that does not appear as a weak immersion in any graph in $\mathcal{G}$.

(ii) There exists a graph F of maximum degree 4 that does not appear as a strong immersion in any graph in $\mathcal{G}$.

(iii) There exists an integer $t \geq 4$ such that every graph in $\mathcal{G}$ has tree-width at most t and maximum degree at most t.

Let us remark that the assumption that graphs in $\mathcal{G}$ are simple is important—if $\mathcal{G}$ is the class of all graphs that can be obtained from paths by replacing each edge by at least four parallel edges, then $\mathcal{G}$ satisfies (ii), but not (i) and (iii). Furthermore, if $\mathcal{G}$ is the class of graphs obtained from simple 4-edge-connected 4-regular graphs of bounded tree-width (say at most 10) by replacing one edge by any positive number of parallel edges, then $\mathcal{G}$ satisfies (i) and (ii), but not (iii). The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) hold even if $\mathcal{G}$ contains non-simple graphs, though. Obviously, in propositions (i) and (ii), we could restrict F to be a square grid.

Flows in networks can be used to determine whether a k-system of large magnitude with a given center and rays exists. This enables us to restate Theorem 2 in the following form.

Theorem 5. *Let G be a graph and c a vertex of G. Let $X \subseteq V(G) \setminus \{c\}$ be any set of vertices such that G contains no edge cut of size less than k separating c from a vertex in X. If a graph F of maximum degree at most $k \geq 3$ does not appear in a graph G as a strong immersion, then there exist sets $Y \subseteq X$ and $K \subseteq E(G)$ such that $k|Y| + |K| < d(k)|V(F)|$ and the component of $G - Y - K$ that contains c does not contain any vertex of X.*

Theorem 5 forms a basis for a structure theorem for strong immersions analogous to the one for weak immersions [4, 18], which we develop in a future paper. Here, let us state just the first step towards this structure.

Theorem 6. *For every graph F and an integer $m \geq 0$, there exists a constant M such that the following holds. Let G be a graph and $X \subseteq V(G)$ a set of its vertices such that no two vertices of X are separated by an edge cut of size less than M in G. Let G_X be the graph with vertex set X in that two vertices $u, v \in X$ are adjacent if G contains m pairwise edge-disjoint paths joining u with v and otherwise disjoint with X. If G does not contain F as a strong immersion, then G_X is connected.*

Let K_F be a multistar with rays $V(F)$, such that each vertex $v \in V(F)$ has the same degree in F and in K_F. Observe that if $m \geq 2|E(F)|$, then the graph G_X cannot contain the star $K_{1,|V(F)|}$ as a minor, as otherwise Menger's theorem would imply that G contains K_F as a strong immersion, and consequently that G contains F as a strong immersion. This restricts the structure of G_X significantly, and to obtain a structure theorem, it remains to argue how the rest of the graph can attach to this well-structured part of G.

In Section 1, we prove Theorems 4, 5 and 6 under assumption that Theorem 2 holds. Section 2 is devoted to the proof of Theorem 2.

1 Corollaries

Proof of Theorem 4. The implication (i) $\Rightarrow$ (ii) is trivial.

Note that for every graph F of maximum degree 4, there exists an integer g such that F is strongly immersed in the $g \times g$ grid. Let t_1 be the constant of Theorem 1 for this g. Let $t_2 = d(4)|V(F)|$, and let $t = \max(t_1, t_2)$. If a graph F of maximum degree 4 does not appear as a strong immersion in a graph $G \in \mathcal{G}$, then by Theorem 1, G has tree-width at most $t_1 \leq t$, and by Corollary 3, G has maximum degree at most $t_2 \leq t$. Therefore, (ii) $\Rightarrow$ (iii) holds.

Suppose now that (iii) holds, i.e., there exists an integer $t \geq 4$ such that every graph $G \in \mathcal{G}$ has tree-width at most t and maximum degree at most t. Let F be a sufficiently large 4-regular expander (see [10] for a construction); say, for some $\varepsilon > 0$, $|V(F)| \geq 5t + 5 + \frac{3(t+1)^2}{\varepsilon}$ and for each $S \subseteq V(F)$ of size at most $|V(F)|/2$, there are at least $\varepsilon|S|$ edges in F between S and $V(F) \setminus S$. We claim that F does not appear as a weak immersion in G.

Suppose on the contrary that θ is an immersion of F in G, and let $T = \theta(V(F))$. Since G has tree-width at most t, there exist sets $A, B \subset V(G)$ such that $A \cup B = V(G)$, no edge of G joins a vertex in $A \setminus B$ with a vertex in $B \setminus A$, $|A \cap B| \leq t+1$, $|A \cap T| \geq \frac{1}{3}(|T|-2t-2)$ and $|B \cap T| \geq \frac{1}{3}(|T|-2t-2)$. Without loss of generality, assume that $|T \setminus B| \leq |T \setminus A|$. Let $S_A = \theta^{-1}(T \setminus B)$ and $S_B = \theta^{-1}(T \cap B) = V(F) \setminus S_A$. Note that $\frac{1}{3}(|V(F)| - 5t - 5) \leq |S_A| \leq |V(F)|/2$. Let Z be the set of edges of F between S_A and S_B; we have $|Z| \geq \varepsilon|S_A|$. Consequently, the subgraph $Q = \bigcup_{e \in Z} \theta(e)$ contains at least $\varepsilon|S_A|$ edges incident with vertices of $A \cap B$, and at least one vertex of $A \cap B$ has degree at least $\varepsilon|S_A|/(t+1) \geq t+1$. This contradicts the assumption that the maximum degree of G is at most t, showing that (iii) $\Rightarrow$ (i) holds. $\square$

Proof of Theorem 5. Since G does not contain F as a strong immersion, Theorem 2 implies that G does not contain a k-system of magnitude $d(k)|V(F)|$. Let G' be the network with the vertex set $V(G) \cup \{z\}$, where z is a new vertex not appearing in $V(G)$, and the edge set defined as follows: For each edge $e \in E(G)$ not incident with a vertex in X, add a pair of edges in opposite directions joining the endvertices of e. For each edge $e \in E(G)$ joining a vertex $u \notin X$ with a vertex $v \in X$, add an edge directed from u to v. For each vertex $x \in X$, add an edge directed from x to z. The edges incident with z have capacity k, while all other edges of the network have capacity 1. If G' contained a flow of size $d(k)|V(F)|$ from c to z, then the corresponding pairwise edge-disjoint paths in G would form a k-system of magnitude $d(k)|V(F)|$, as each vertex of X is contained in at most k such paths. Consequently, no such flow exists.

By the flow-cut duality, it follows that G' contains an edge cut K' of capacity less than $d(k)|V(F)|$ separating c from z. Let Y be the set of vertices $y \in X$ such that the edge xz belongs to K'. Let K be the set of edges of G corresponding to the edges of K' not incident with z. Clearly, $G - Y - K$ contains no path from c to a vertex of X. Furthermore, $k|Y| + |K|$ is equal to the capacity of K', and thus $k|Y| + |K| < d(k)|V(F)|$. $\square$

Proof of Theorem 6. Let $k = \max(\Delta(F), 3)$, $s = d(k)|V(F)|$ and $M = ms^3 + s^2$. Suppose that a graph G and a set $X \subseteq V(G)$ satisfy the assumptions of Theorem 6. Consider any nonempty disjoint sets $A, B \subset X$ such that $A \cup B = X$. Let c_A be an arbitrary vertex of A and apply Theorem 5 for c_A and B, obtaining sets $Y_B \subseteq B$ and $K_B \subseteq E(G)$, where $k|Y_B| + |K_B| < s$, such that the component of

$G-K_B-Y_B$ that contains c_A does not contain any vertex of B. For each $y \in Y_B$, apply Theorem 5 for y and A, obtaining sets $Y_A^y \subseteq A$ and $K_A^y \subseteq E(G)$, where $k|Y_A^y|+|K_A^y|<s$, such that the component of $G-K_A^y-Y_A^y$ that contains y does not contain any vertex of A. Let $K=K_B\cup\bigcup_{y\in Y_B} K_A^y$ and let $Y_A=\bigcup_{y\in Y_B} Y_A^y$, and note that $|K|\le s^2$ and $|Y_A|\le s^2$.

Let c_B be an arbitrary vertex of B. By Menger's theorem, there exists a set $\mathcal{P}_0$ of M pairwise edge-disjoint paths from c_A to c_B in G. Let $\mathcal{P}\subseteq\mathcal{P}_0$ consist of the paths that do not contain edges of K; we have $|\mathcal{P}|\ge M-s^2$. Consider a path $P\in\mathcal{P}$. Let v_0, v_1, $\ldots$, v_p be the vertices of P in order, where $v_0=c_A$ and $v_p=c_B$. Let $j>0$ be the smallest index such that v_j belongs to B. As the component of $G-K_B-Y_B$ that contains c_A does not contain any vertex of B, the vertex v_j belongs to Y_B. Let i be the largest index such that $i<j$ and v_i belongs to A. As the component of $G-K_A^{v_j}-Y_A^{v_j}$ that contains v_j does not contain any vertex of A, it follows that v_i belongs to $Y_A^{v_j}\subseteq Y_A$. Consequently, G contains a set of $|\mathcal{P}|$ pairwise edge-disjoint paths joining vertices of Y_A with vertices of Y_B and otherwise disjoint from X. By the pigeonhole principle, there exist vertices $a\in A$ and $b\in B$ incident with at least $\frac{|\mathcal{P}|}{|Y_A||Y_B|}\ge m$ of these paths, and thus ab is an edge of G_X.

Therefore, for all nonempty disjoint sets $A,B\subset X=V(G_X)$ such that $A\cup B=V(G_X)$, there exists an edge between a vertex of A and a vertex of B in G_X. It follows that G_X is connected. □

2 Proof of Theorem 2

We need the following variation on the Mader's splitting theorem [11]. Let G be a graph, x a vertex of G and for each $s,t\in V(G)\setminus\{x\}$, let $\lambda(s,t)$ denote the maximum number of pairwise edge-disjoint paths between s and t in G. Let e and f be edges joining x to vertices u and v, respectively, and let G' be the graph obtained from $G-\{e,f\}$ by adding a new edge joining u with v. We say that the pair of edges e and f is *splittable* if for every $s,t\in V(G')\setminus\{x\}$, the graph G' contains $\lambda(s,t)$ pairwise edge-disjoint paths between s and t. We say that G' is obtained by *lifting* the edges e and f. Note that G' is immersed in G.

Theorem 7 (Frank [6])**.** *Let G be a graph and let x be a vertex of G not incident with any 1-edge cut. If x has degree $m\neq 3$, then there are $\lfloor m/2\rfloor$ pairwise disjoint splittable pairs of edges incident with x.*

If (H,σ) is a k-system and σ is slim, we consider the paths in $\sigma(E(H))$ to be directed away from $\sigma(c)$, where $c=c(H)$. That is, if $e=cv$ is an edge of H, then $\sigma(c)$ is the first vertex of $\sigma(e)$ and $\sigma(v)$ is the last vertex of $\sigma(e)$.

Definition 1. *We say that a triple (G,H,σ) is* well-behaved *if (H,σ) is a k-system of magnitude d in a graph G, such that σ is slim and the following conditions are satisfied:*

(w1) *G is 3-edge connected,*

(w2) *vertices in $V(G)\setminus V(\sigma)$ have degree exactly 3,*

(w3) *for every $v\in R(H)$, if K is an k-edge cut in G separating $\sigma(c(H))$ from $\sigma(v)$, then K consists of the edges incident with $\sigma(v)$,*

(w4) *every edge of $E(G)\setminus E(\sigma)$ is incident with a vertex in $\sigma(R(H))$ of degree exactly k,*

(w5) *for each vertex $v\in V(\sigma)\setminus\sigma(V(H))$, at most one edge incident with v does not belong to $E(\sigma)$. Furthermore, if there is such an edge and v belongs to the path $\sigma(e)$ for an edge $e\in E(H)$, then v is the next-to-last vertex of $\sigma(e)$ and the last vertex of $\sigma(e)$ has degree exactly k.*

Let H be a multistar with a strong immersion σ in a graph G and let H' be a multistar with a strong immersion σ' in a graph G'. We say that (G',H',σ') is a *reduction* of (G,H,σ) if there exists a weak immersion θ of G' in G satisfying the following conditions:

- $\theta(\sigma'(c(H')))=\sigma(c(H))$,

- $\theta(\sigma'(R(H'))) \subseteq \sigma(R(H))$, and
- if $e \in E(G')$ is not incident with a vertex $v \in \sigma'(V(H'))$, then $\theta(e)$ does not contain $\theta(v)$.

Note that if a strong immersion α of $S_{n,k}$ in G' respects (H', σ'), then $\theta \circ \alpha$ is a strong immersion of $S_{n,k}$ in G respecting (H, σ). Furthermore, the reduction relation is transitive.

Lemma 8. *If (H_0, σ_0) is a k-system of magnitude d in a graph G_0, then there exists a reduction (G, H, σ) of (G_0, H_0, σ_0) such that (H, σ) is a k-system of magnitude d and (G, H, σ) is well-behaved.*

Proof. Let G with a k-system (H, σ) of magnitude d be chosen so that (G, H, σ) is a reduction of (G_0, H_0, σ_0), and subject to that $|V(G)| + |E(G)| + |E(\sigma)|$ is minimal. Such triple (G, H, σ) exists, since (G_0, H_0, σ_0) is a reduction of itself. Clearly, σ is slim and G has no loops. We claim that (G, H, σ) is well-behaved. Let us discuss the conditions **(w1)**, …, **(w5)** separately.

(w1) Suppose that there is an edge cut K of size at most 2 in G; we can assume that K is minimal. Since $k \geq 3$, **(s3)** implies that all vertices of $\sigma(V(H))$ are in the same component C of $G - K$. Let $G' = C$ if $|K| \leq 1$; let G' be the graph obtained from C by adding an edge e (possibly a loop) between the vertices in C incident with K if $|K| = 2$. Let $\sigma'(v) = \sigma(v)$ for $v \in V(H)$. Consider $f \in E(H)$. If $\sigma(f)$ contains two edges of K, then let $\sigma'(f) = (\sigma(f) \cap C) + e$, otherwise let $\sigma'(f) = \sigma(f) \cap C$. Observe that (G', H, σ') is a reduction of (G, H, σ). This is a contradiction, as (G, H, σ) was chosen so that $|V(G)| + |E(G)| + |E(\sigma)|$ is minimal.

(w2) Suppose G satisfies **(w1)** and contains a vertex $v \in V(G) \setminus V(\sigma)$ of degree greater than 3. By Theorem 7, we can lift a pair of edges incident to v without violating the condition **(s3)**. This way, we obtain a reduction (G', H, σ') contradicting the minimality of (G, H, σ).

(w3) Suppose G contains an edge cut K of size k separating $\sigma(c(H))$ from $\sigma(v)$ for some $v \in R(H)$, such that K does not consist of the edges incident with $\sigma(v)$. By **(s3)**, $G - K$ has only two components. Let G' be the graph obtained from G by replacing the component C of $G - K$ that contains $\theta(v)$ by a single vertex w of degree k, incident with the edges of K. Let $Z \subseteq E(H)$ be the set of edges $e \in E(H)$ such that $\sigma(e)$ contains an edge of K. Let H' be the multistar obtained from H by making all edges of Z incident with v instead of their original incident ray and removing all resulting isolated vertices. Let σ' be the strong immersion of H' in G' such that $\sigma'(v) = w$, $\sigma'(x) = \sigma(x)$ for $x \in V(H') \setminus \{v\}$, $\sigma'(e) = \sigma(e)$ for $e \in E(H) \setminus Z$, and $\sigma'(e)$ is the segment of the path $\sigma(e)$ between $\sigma'(c(H'))$ and the first edge of K appearing in the path (inclusive). Note that w has degree exactly k in H', and thus H' satisfies **(s2)**. Furthermore, if $x \in R(H')$, then $\sigma'(x)$ is not separated from $\sigma(c(H'))$ by an edge cut K' of size less than k, as otherwise K' (with the edges incident with w replaced by the corresponding edges of K) separates $\sigma(x)$ from $\sigma(c(H))$; hence, **(s3)** holds for (H', σ'). Also, by **(s3)** for (H, σ), there exist k pairwise edge-disjoint paths in G joining $\sigma(v)$ with the edges of K. Consequently, there exists a strong immersion of G' in G, showing that (G', H', σ') is a reduction of (G, H, σ). This contradicts the minimality of the latter.

(w4) Suppose that G satisfies **(w3)** and $e \in E(G) \setminus E(\sigma)$ is not incident with a vertex of $\sigma(R(H))$ of degree exactly k. Consequently, e is not contained in any k-edge cut separating $\sigma(c(H))$ from a vertex in $\sigma(R(H))$, and we conclude that $(G - e, H, \sigma)$ is a reduction contradicting the minimality of (G, H, σ).

(w5) Suppose that G satisfies **(w1)** and **(w4)** and consider a vertex $v \in V(\sigma) \setminus \sigma(V(H))$. If at least two edges incident with v do not belong to $E(\sigma)$, then by Theorem 7, there exists a splittable pair of edges e and f incident with v such that $e \notin E(\sigma)$. Let u and w be the endvertices of e and f, respectively, distinct from v. Let e' be an edge incident with v and not belonging to $\{e\} \cup E(\sigma)$, and let z be the vertex incident with e' distinct from v. By **(w4)**, u and z are vertices of $\sigma(R(H))$ of degree exactly k. Let G' be the graph obtained from G by lifting the

edges e and f, creating a new edge h. If $f \notin E(\sigma)$, then let $H' = H$ and $\sigma' = \sigma$. Otherwise, consider the edge $f_0 \in E(H)$ such that $f \in E(\sigma(f_0))$. If w appears before v in $\sigma(f_0)$, then let H' be the graph obtained from H by making f_0 incident with $\sigma^{-1}(u)$ instead of its original incident ray (and possibly removing the resulting isolated vertex) and let σ' be obtained from σ by letting $\sigma'(f_0)$ consist of h and the subpath of $\sigma(f_0)$ between $\sigma(c(H))$ and w. If w appears after v in $\sigma(f_0)$, then let H' be the graph obtained from H by making f_0 incident with $\sigma^{-1}(z)$ instead of its original incident ray (and possibly removing the resulting isolated vertex) and let σ' be obtained from σ by letting $\sigma'(f_0)$ consist of e' and the subpath of $\sigma(f_0)$ between $\sigma(c(H))$ and v. Note that (G', H', σ') is a reduction contradicting the minimality of (G, H, σ).

Let us now consider the case that $v \in V(\sigma) \setminus \sigma(V(H))$ is incident with exactly one edge e not belonging to $E(\sigma)$, where e joins v with a vertex u. Note that u belongs to $\sigma(R(H))$ and has degree exactly k. Let f_0 be an edge of H such that $\sigma(f_0)$ contains v. Let H' be obtained from H by making f_0 incident with $\sigma^{-1}(u)$ instead of its original incident ray (and possibly removing the resulting isolated vertex) and let σ' be obtained from σ by letting $\sigma'(f_0)$ consist of e and the subpath of $\sigma(f_0)$ between $\sigma(c(H))$ and v. Note that (G, H', σ') is a reduction of (G, H, σ). By the minimality of (G, H, σ), we have that v is the next-to-last vertex of the path $\sigma(f_0)$. Furthermore, if the last vertex x of $\sigma(f_0)$ had degree greater than k, then we could find a reduction of (G, H', σ') contradicting the minimality of (G, H, σ) in the same way as in the proof of **(w3)** or **(w4)**.

□

Lemma 9. *If (H, σ) is a k-system of magnitude d in a graph G, then there exists a reduction (G', H', σ') of (G, H, σ) such that (H', σ') is a k-system of magnitude $\frac{d}{k(k+1)}$ in G', (G', H', σ') is well-behaved and each vertex of $\sigma'(R(H'))$ has degree exactly k in G'.*

Proof. By Lemma 8, we can assume that (G, H, σ) is well-behaved. Let S be the set of all vertices $s \in R(H)$ such that $\sigma(s)$ has degree exactly k in G and let $B = R(H) \setminus S$. Since σ is slim, **(s2)** and **(s3)** imply that for each $v \in B$, there exists an edge $e \in E(G) \setminus E(\sigma)$ incident with $\sigma(v)$. By **(w4)**, e is incident with a vertex in $\sigma(S)$. We conclude that $|B| \leq k|S|$. Since $|B| + |S| = |R(H)|$, we have $|S| \geq |R(H)|/(k+1)$. Let $H_0 = H - B$. Since H is connected and satisfies **(s2)**, we have $|E(H_0)| \geq |S| \geq |R(H)|/(k+1) \geq \frac{d}{k(k+1)}$. Let $\sigma_0 = \sigma \upharpoonright (V(H_0) \cup E(H_0))$. Clearly, (H_0, σ_0) is a k-system of magnitude $\frac{d}{k(k+1)}$ in G, (G, H_0, σ_0) is a reduction of (G, H, σ) and each vertex of $\sigma_0(R(H_0))$ has degree exactly k in G. Finally, we obtain a well-behaved reduction (G', H', σ') by Lemma 8, since no vertices of degree greater than k belonging to $\sigma'(R(H'))$ are created in its proof. □

Let (G, H, σ) be well-behaved, where (H, σ) is a k-system of magnitude d in a graph G such that each vertex of $\sigma(R(H))$ has degree exactly k. We can assume that all edges of G between $\sigma(c)$ and $\sigma(R(H))$ belong to $E(\sigma)$, as otherwise we can add more edges to H. Let $N(\sigma)$ consist of $\sigma(R(H))$ and of all vertices incident with edges of $E(G) \setminus E(\sigma)$. Let $M(\sigma) = N(\sigma) \cap V(\sigma) \setminus \sigma(R(H))$. Let us note that by **(w5)**, $M(\sigma)$ is an independent set in G. Consequently, $\sigma(e)$ intersects $N(\sigma)$ in at most two vertices for each $e \in E(H)$. Let G' be the graph with vertex set $\sigma(c(H)) \cup N(\sigma)$ and the edge set defined as follows: the subgraphs of G and G' induced by $N(\sigma)$ are equal; and, the edges incident with $\sigma(c(H))$ are $\{f_e : e \in E(H)\}$, where f_e joins $\sigma(c(H))$ with the first vertex of $\sigma(e)$ that belongs to $N(\sigma)$. Let us define a strong immersion σ' of H in G' as follows: For $v \in V(H)$, we set $\sigma'(v) = \sigma(v)$. For $e \in E(H)$, let $\sigma'(e)$ consist of f_e and of $\sigma(e) \cap G[N(\sigma)]$. Note that $\sigma'(e)$ has length at most two. Clearly, (G', H, σ') is a reduction of (G, H, σ). We say that (G', H, σ') is a *core* of (G, H, σ).

Let us define a function $g : E(G') \to E(G)$ as follows: if $f \in E(G')$ is not incident with $\sigma(c(H))$, then let $g(f) = f$. Otherwise, $f = f_e$ for some $e \in E(H)$, and we let $g(f)$ be equal to the last edge of $\sigma(e)$ that does not belong to $G[N(\sigma)]$. We say that g is the *origin function* of the core.

Lemma 10. *Let (G, H, σ) be well-behaved, where (H, σ) is a k-system of magnitude d in a graph G such that each vertex of $\sigma(R(H))$ has degree exactly k. If (G', H, σ') is the core of (G, H, σ), then (H, σ') is a k-system of magnitude d in G'.*

Proof. It suffices to check that (G', H, σ') satisfies the condition **(s3)**. Let g be the origin function of (G', H, σ'). Consider an edge cut K in G' separating $\sigma'(c(H))$ from $\sigma'(v)$ for some $v \in R(H)$. Observe that $g(K)$ is an edge cut in G separating $\sigma(c(H))$ from $\sigma(v)$, and thus $|K| \geq |g(K)| \geq k$ by **(s3)** for (G, H, σ). □

Let (H, σ) be a k-system of magnitude d in a graph G. We say that (G, H, σ) is *peeled* if it is well-behaved, each vertex of $\sigma(R(H))$ has degree exactly k, all edges incident with $\sigma(c(H))$ belong to $E(\sigma)$ and $V(G) = \{\sigma(c(H))\} \cup N(\sigma)$. Note that every core is peeled.

Lemma 11. *Let (H_0, σ_0) be a k-system of magnitude d in a graph G_0. Then there exists a reduction (G, H, σ) of (G_0, H_0, σ_0) that is peeled, the k-system (H, σ) has magnitude $\frac{d}{k(k+1)}$ and no vertex of G other than $\sigma(c(H))$ has degree greater than $2k + 1$.*

Proof. Let (G, H, σ) be a peeled reduction of (G_0, H_0, σ_0), where the k-system (H, σ) has magnitude $\frac{d}{k(k+1)}$, with $|E(G)|$ as small as possible (which exists by Lemmas 9 and 10). Suppose that a vertex $v \in V(G)$ has degree at least $2k + 2$ and $v \neq \sigma(c(H))$. By **(w2)** and the assumption that (G, H, σ) is peeled, we have $v \in M(\sigma)$. Note that v is joined with $\sigma(c(H))$ by at least $k + 1$ edges. Select an arbitrary edge $e \in E(H)$ such that $\sigma(e)$ contains v and let f_1 and f_2 be the edges of $\sigma(e)$, where f_1 is incident with $\sigma(c(H))$. Let G' be the graph obtained from G by lifting f_1 and f_2, creating a new edge f. Let σ' be obtained from σ by letting $\sigma'(e)$ be the path consisting only of f. Note that (G', H, σ') is a reduction of (G, H, σ).

We claim that (H, σ') is a k-system in G'. It suffices to check that it satisfies the condition **(s3)**. Let K' be a minimal edge cut in G' separating $\sigma'(c(H))$ from $\sigma'(x)$ for some $x \in R(H)$. Let C and X be the vertex sets of the components of $G' - K'$, where C contains $\sigma'(c(H))$. Let K be the set of edges between C and X in G. If K does not contain f_1, then $|K'| = |K|$, and thus $|K'| \geq k$ by **(s3)** for (G, H, σ). If K contains f_1, then K also contains all edges parallel to f_1, and these edges belong to K' as well. Since v and $\sigma(c(H))$ are joined by at least $k + 1$ edges, we have $|K'| \geq k$. We conclude that (G', H, σ') satisfies the condition **(s3)**.

Note that (G', H, σ') is peeled, and thus it contradicts the minimality of (G, H, σ). □

Lemma 12. *Let $k \geq 3$ be an integer and let (H, σ) be a k-system in a graph G, where (G, H, σ) is peeled. For each $v \in \sigma(R(H))$, there exists a set of k pairwise edge-disjoint paths in G, each of length at most $4k + 2$, joining $\sigma(v)$ with $\sigma(c(H))$.*

Proof. By **(s3)**, there exist k pairwise edge-disjoint paths $Q_1, \ldots, Q_k$ between $\sigma(v)$ and $\sigma(c(H))$; let $Q = Q_1 \cup \ldots \cup Q_k$ and let S be the set of edges $e \in E(H)$ such $\sigma(e) \subseteq Q_i$ for some $i \in \{1, \ldots, k\}$. Let us choose these paths so that $|E(Q)|$ is as small as possible, and subject to that $|S|$ is as large as possible. Clearly, Q contains exactly k edges incident with $\sigma(c(H))$.

Let e be an edge of H such that $\sigma(e)$ shares at least one edge f with Q, say $f \in E(Q_1)$. Suppose that f is not incident with $\sigma(c(H))$, and let f' be the other edge of $\sigma(e)$. If f' were not in $E(Q)$, then we could change Q_1 to use f' to enter $\sigma(c(H))$, thus either decreasing $|E(Q)|$, or adding e to S, contrary to the choice of the paths $Q_1, \ldots, Q_k$. We conclude that if $\sigma(e)$ contains an edge of Q, then $Q \cap \sigma(e)$ contains an edge incident with $\sigma(c(H))$. Consequently, there are at most k such edges $e \in E(H)$. Let

$$W = \bigcup_{e \in E(H), E(\sigma(e)) \cap E(Q) \neq \emptyset} V(\sigma(e)) \setminus \{\sigma(c(H))\}.$$

Since $\sigma(e)$ has length at most two for each $e \in E(H)$, we have $|W| \leq 2k$.

Consider the path Q_i for some $i \in \{1, \ldots, k\}$, and let $v_0 v_1 \ldots v_\ell$ be its vertices in order, where $v_0 = \sigma(c(H))$ and $v_\ell = v$. Let Z be the set of vertices of Q_i belonging to $\sigma(R(H)) \cup M(\sigma)$. Suppose

that there exists a vertex $z \in Z \setminus W$. Note that there exists an edge $e \in E(H)$ such that z belongs to $\sigma(e)$, and by the definition of W, we have $E(\sigma(e)) \cap E(Q) = \emptyset$. Therefore, we can change the path Q_i to follow $\sigma(e)$ from z to $\sigma(c(H))$. Since we have chosen the paths with $|E(Q)|$ as small as possible, we conclude that the distance from z to $\sigma(c(H))$ in Q_i is at most two. Consequently, $(Z \setminus W) \cap V(Q_i) \subseteq \{v_1, v_2\}$. Since (G, H, σ) is peeled and satisfies **(w4)**, every edge of Q_i is incident with a vertex of Z, and thus each edge of $Q_i - \{v_0, v_1, v_2\}$ is incident with a vertex of $Z \cap W$. Since $|W| \leq 2k$ and $v_\ell = v \in Z$, this implies that Q_i has length at most $4k + 2$. □

Proof of Theorem 2. Let $N = (2k+1)^{8k+4}$ and $d = d(k)n = k^2(k+1)Nn$. By Lemma 11, there exists a peeled reduction (G', H', σ') of (G, H, σ), where (H', σ') is a k-system of magnitude $d' = \frac{d}{k(k+1)}$ in G' and no vertex of G' other than $\sigma'(c(H'))$ has degree greater than $2k+1$. Consequently, for each $v \in \sigma'(R(H'))$, there exist at most N vertices at distance at most $8k+4$ from v in $G' - \sigma'(c(H'))$. Since $|R(H')| \geq d'/k \geq Nn$, we can greedily choose a set $U \subset \sigma'(R(H'))$ of size n such the distance in $G' - \sigma'(c(H'))$ between any two vertices of U is at least $8k+5$. By Lemma 12, we can for each $u \in U$ find a set S_u of k pairwise edge-disjoint paths in G' joining u with $\sigma'(c(H'))$, each of length at most $4k+2$. By the choice of U, the paths $\bigcup_{u \in U} S_u$ are pairwise edge-disjoint and intersect only in their endvertices. This set of paths corresponds to a strong immersion of $S_{n,k}$ in G' respecting (H', σ'). We conclude that G contains a strong immersion of $S_{n,k}$ respecting (H, σ). □

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